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THE EFFECT OF A VARIABLE SCALE HEIGHT ON DETERMINATIONS OF ATMOSPHERIC DENSITY FROM SATELLITE ACCELERATIONS

by

Luigi G. Jacchia [1]

Summary. Formulas for the determination of atmospheric densities from satellite accelerations assume that the scale height is constant at heights immediately above perigee. The error of these formulas is evaluated for an atmosphere in which, from a representative value at perigee, the density scale height increases linearly with height at different rates.

The anomalistic period P of an artificial satellite changes under the action of atmospheric drag. The rate of change can be expressed very approximately (Sterne, 1958a; King-Hele, Cook and Walker, 1959) by the equation,

$$\frac{dP}{dt} = -3\, C_D \frac{A}{m}\, a \int_0^{\pi} \frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}}\, \rho\, dE \quad . \tag{1}$$

The symbols are defined as follows:

C_D = drag coefficient,

A = effective cross-section of satellite,

m = mass of satellite,

a = semimajor axis of satellite's orbit,

e = orbital eccentricity of satellite,

E = eccentric anomaly,

ρ = atmospheric density.

Equation (1) assumes a stationary atmosphere; the effect of atmospheric rotation has been evaluated by Sterne (1959). The only approximation made in equation (1) is that the orbit can be defined by a set of Keplerian elements in the course of one revolution; in the general case, therefore, the error arising from this approximation is entirely negligible.

Useful formulas for computing atmospheric densities from satellite accelerations can be derived from equation (1) on the assumption of a spherically symmetrical atmosphere in which the density varies exponentially with height (Sterne, 1958b; Groves, 1958; King-Hele, Cook and Walker, 1959). The procedure consists in replacing ρ by the expression,

$$\rho = \rho_p \exp\left(-\frac{r - q}{H}\right) \quad , \tag{2}$$

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(where ρ_p is the atmospheric density at perigee; r the geocentric distance; q the perigee distance; and H a constant, the density scale height), and in expanding the integrand as a power series of e. The integral can then be evaluated in terms of Bessel functions of the first kind of imaginary argument $I_n(x)$, with $x = ae/H$. For small eccentricities, the Bessel functions can be replaced by their expansion at the origin; for larger eccentricities, use is made of their asymptotic expansions (King-Hele, Cook and Walker, 1959).

Formulas of the type just described have been used by all investigators in deriving atmospheric densities from satellite accelerations. If the density scale height H varies with height, instead of being a constant, the use of such formulas causes a systematic error. It is the purpose of this note to evaluate the size of this error.

Let us assume, for simplicity, that H varies linearly with height, and that H_p is the value of H at perigee; we shall then have

$$H = H_p + \beta\,(r - q) \quad ; \tag{3}$$

and, since by definition

$$\frac{1}{H} = -\frac{1}{\rho}\,\frac{d\rho}{dr} \quad ,$$

we obtain, by integration,

$$\rho = \rho_p \left[1 + \frac{\beta}{H_p}\,(r - q) \right]^{-\frac{1}{\beta}} \tag{4}$$

The gradient β of the density scale height is a non-dimensional quantity. Recent atmospheric results (Jacchia, 1960) have shown that at the height of 400 km $\beta \approx +0.1$ in the dark hemisphere and $\beta \approx +0.2$ in the center of the diurnal bulge. The density scale height itself is of the order of 55 km at night and 72 km in the diurnal bulge, for the same height of 400 km above the geoid.

Replacing ρ in equation (1) by the expressions of equations (2) and (4) respectively, and using the same values for ρ_p and H_p in both cases, we obtain two values of $\frac{d\rho}{dt}$, whose ratio R is 1 for $\beta = 0$ and >1 for positive values of β. The explicit value of R is:

$$R = \frac{\displaystyle\int_0^{\pi} f(e,E)\,\exp\left(-\frac{r-q}{H_p}\right) dE}{\displaystyle\int_0^{\pi} f(e,E) \left[1 + \frac{\beta}{H_p}\,(r - q) \right]^{-\frac{1}{\beta}} dE} \quad , \tag{5}$$

where

$$f(e,E) = \frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}}$$

It should be apparent that R - 1 is the relative error that is committed when ρ_p is determined from equation (1) under the assumption that $\beta = 0$.

The values of R to be found in Table 1 were computed by numerical evaluation of the integrals of equation (5), with the value, $H_p = 0.01$ earth's radii = 63.78 km. Extensive use was made of the "logarithmic" finite-difference method of integration introduced by the author (Jacchia, 1955), which is highly advantageous when the integrand is a near-exponential function.

It had been assumed at the outset that for values of H_p different from 63.78 km, R must differ from the values given in Table 1. Sample integrations with $H_p = 31.89$ km seem, however, to reproduce Table 1 to the last digit. The reason for this fact is not obvious to this writer. No serious attempt has been made to prove that $\partial R / \partial H_p = 0$; the problem is hereby left to an investigator endowed with greater persistence or deeper mathematical insight.

It will be noticed that, while R - 1 is nearly proportional to β for any given value of e, its behavior is quite different when its variation in function of e is considered for a given β (figures 1 and 2). For e = 0, we must obviously have $R \equiv 1$, irrespective of β; even an extremely small eccentricity, however, will make R considerably different from unity when $\beta \neq 0$, and for any given value of β, R reaches a maximum for $e \approx 0.02$. For greater values of e, R becomes a little smaller and rapidly approaches a nearly asymptotic value which is practically reached for e = 0.2.

It is a pleasure to acknowledge the expert help of Miss J. R. B. Carmichael, who performed most of the numerical integrations.

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Table 1

Values of R for $H_p = 63.78$ km

e	Values of R $\beta = 0$	$\beta = .1$	$\beta = .2$
0.00	1.000	1.000	1.000
0.01	1.000	1.032	1.061
0.02	1.000	1.050	1.100
0.05	1.000	1.044	1.095
0.10	1.000	1.040	1.086
0.20	1.000	1.039	1.083
0.40	1.000	1.039	1.082
0.60	1.000	1.039	1.082
1.00	1.000	1.039	1.083

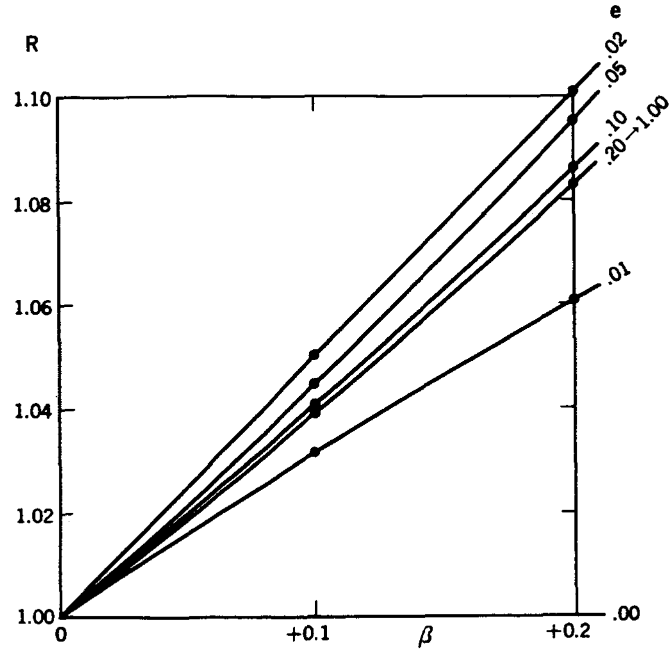

FIGURE 1. - Variation of R (see equation (5)) in function of the scale-height gradient β for different orbital eccentricities. The relative error of densities determined using formulae that assume $\beta = 0$ is given by $R - 1$. Scale height at perigee, $H_p = 63.78$ km.

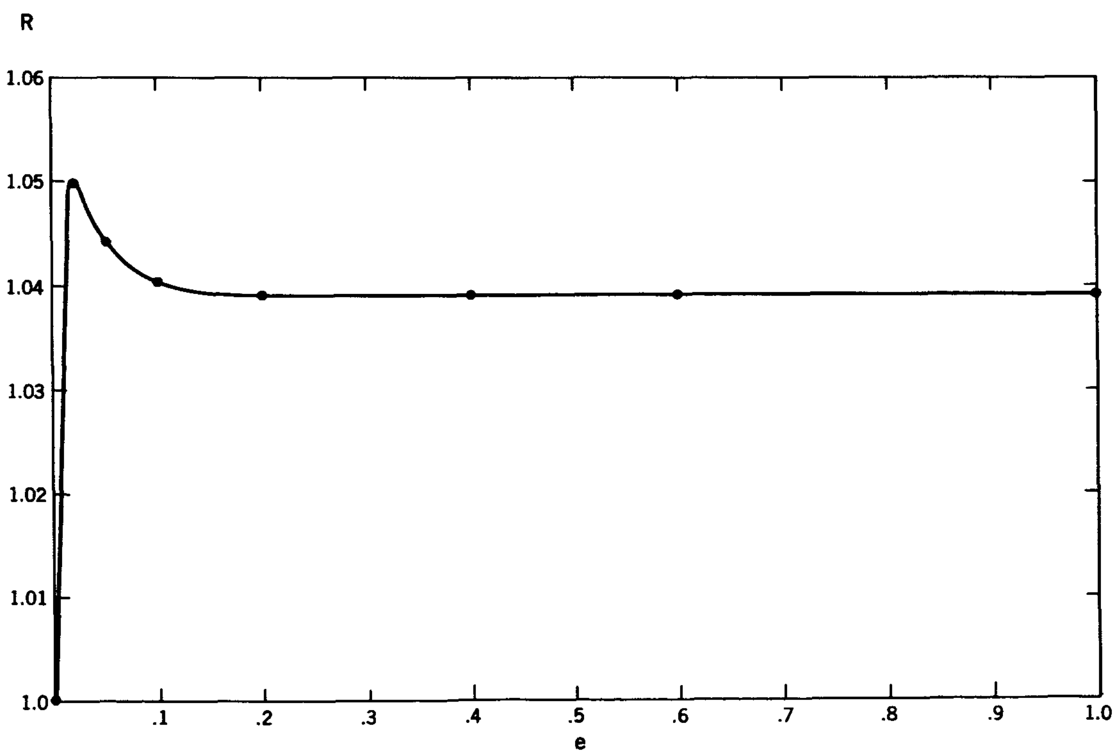

FIGURE 2. - Variation of R in function of e for $\beta = +0.1$. (For R see legend to Fig. 1.) Scale height at perigee, $H_p = 63.78$ km.

COMMENT ON THE PAPER ENTITLED "SYMMETRY OF THE EARTH'S FIGURE" [1]

by

Charles A. Whitney [2]

Summary. A recent criticism of work demonstrating the existence of an appreciable third harmonic in the earth's field is shown to be invalid.

From an analysis of the motion of Vanguard I, O'Keefe, Eckles, and Squires (1959) demonstrated the existence of an appreciable third harmonic in the earth's potential field. This claim has recently been criticized by Brenner, Fulton, and Sherman (1960). Unfortunately, however, these authors have misunderstood the earlier analysis; their arguments are therefore not relevant and in no sense cast doubt on the existence of the third harmonic.

Brenner et al. have performed numerical integrations of the equations of motion for a drag-free orbit in an oblate potential field lacking a third harmonic. In their Figures 3 and 4 they display the time variations of the osculating values of eccentricity which vary by .4 percent at apogee, .2 percent at perigee and .3 percent at the ascending crossing of latitude +30 degrees. The authors suggest that these may be the variations reported, which were misinterpreted as being evidence for the third harmonic.

The fallacy of their argument is simply stated. The variations they display are produced by the well-known short-period perturbations due to the earth's oblateness. But O'Keefe et al. (1959) have accounted for the effects of these perturbations to a precision that is entirely adequate for the purpose. Therefore the criticism offered does not apply to their analysis.

It is significant that, in an independent analysis of the motions of several additional satellites, using optical as well as Minitrack data, Kozai (unpublished) has not only confirmed the analysis of O'Keefe et al.; he has also been able to improve the earlier value for the coefficient of the third harmonic.

We may demonstrate quantitatively that the supposed long-period terms displayed by Brenner et al. are in reality manifestations of the short-period terms due to oblateness, coupled with the secular advance of perigee. Kozai (1959) has given the following formula for the short-period perturbations of eccentricity due to oblateness:

$$\begin{aligned} de = \frac{1-e^2}{e}\frac{J}{a^2}\Bigg[\frac{1}{3}\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(\frac{a}{r}\right)^3 - (1-e^2)^{-3/2}\right\} \\ + \frac{1}{2}\left(\frac{a}{r}\right)^3 \sin^2 i \cos 2(v+\omega)\Bigg] \\ - \frac{\sin^2 i}{2e}\frac{J}{ap}\left\{\cos 2(v+\omega) + e\cos(v+2\omega) + \frac{1}{3} e\cos(3v+2\omega)\right\} \end{aligned}$$

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[1] To be published in the Journal of the American Rocket Society.

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Not having available the exact numerical values corresponding to the integrations performed by Brenner et al., we adopt for Vanguard I the following mean values:

$$
\begin{aligned}
a &= 1.361 \text{ earth radii} \\
e &= .190 \\
i &= 34\overset{\circ}{.}25 \\
J &= .001624
\end{aligned}
$$

Introducing these into Kozai's equation gives, for the variation of osculating eccentricity at perigee de(0), and at apogee de(π), the following formulae:

$$de(0) = .00064 + .00038 \cos 2\omega ,$$

$$de(\pi) = -.00036 - .00015 \cos 2\omega$$

These formulae reproduce the variations displayed by Brenner, Fulton and Sherman.

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This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory. First issued to ensure the immediate dissemination of data for satellite tracking, the Reports have continued to provide a rapid distribution of catalogues of satellite observations, orbital information, and preliminary results of data analyses prior to publication in the appropriate journals.

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